BRIDGE DISTANCE, HEEGAARD GENUS, AND EXCEPTIONAL SURGERIES

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Abstract. We demonstrate a lower bound on the genus of an essential surface or Heegaard surface in a 3-manifold obtained by non-trivial surgery on a link in terms of the bridge distance of a bridge surface for the link. Consequently, knots with high distance bridge surfaces do not admit non-trivial non-hyperbolic surgeries or non-trivial cosmetic surgeries.

1. Introduction

A non-trivial Dehn surgery on a hyperbolic link L in a connected, compact, orientable 3-manifold M is *exceptional* if it produces either a non-hyperbolic 3-manifold or a manifold homeomorphic to M (in which case the surgery is also called a *cosmetic surgery*.) It has been known since at least Thurston's 2π theorem (improved by Agol and Lackenby [1, 31]) that non-hyperbolic surgeries on a fixed link L are relatively rare, as are cosmetic surgeries on hyperbolic knots [5]. We show that a knot that has a bridge surface with high distance does not admit an exceptional surgery. Many of our techniques and results apply also to links with multiple components in 3-manifolds with boundary; for more details see Corollary 3.2 and Theorem 5.1.

Theorem 6.1. *Let M be a closed, compact, orientable manifold and suppose that $L \subset M$ is a knot in bridge position with respect to a Heegaard surface T_τ for M. Assume that the boundary of a regular neighborhood of L is incompressible in the exterior of L and that if T_τ is a sphere, then $|L \cap T_\tau| \geq 6$. Then:*

(1) If $M = S^3$, $T_\tau = S^2$, and surgery on L produces a reducible 3-manifold, then $d(T_\tau) \leq 2$.
(2) If $M = S^3$, $T_\tau = S^2$, and surgery on L produces a 3-manifold with an essential torus, then $d(T_\tau) \leq 2$.
(3) If $M = S^3$, $T_\tau = S^2$, L and surgery on L produces a lens space, then $d(T_\tau) \leq 3$.

(4) *If $M = S^3$, $T_\tau = S^2$, and surgery on L produces a small Seifert fibered space other than S^3 or a lens space, then $d(T_\tau) \leq 4$.*

(5) *If M does not contain an essential sphere or torus and if L is a hyperbolic knot with a non-trivial non-hyperbolic surgery then $d(T_\tau) \leq 6$.*

(6) *If $M \neq S^3$ and a non-trivial surgery on L produces a 3-manifold with Heegaard genus no larger than that of M, then*

$$d(T_\tau) \leq max(4, 2g(T_\tau) + 2).$$

The bridge distance of a link is defined in Section 2 in terms of the arc and curve complex of a bridge surface for the link and is a slight modification of the definition in [2]. The distance can be thought of as a measure of complexity for the knot or link and Blair, Tomova, and Yoshizawa [6] have constructed knots with arbitrarily high bridge distance. For technical reasons, we assume that if the bridge surface T_τ is a sphere, then the link is not 2–bridge. This is not a severe restriction, since non-hyperbolic surgeries on 2-bridge knots have been classified [8, 19]. The assumption on the incompressibility of the boundary of a regular neighborhood of L is satisfied, for example, if no component of L is unknotted or is the core of a genus 1 Heegaard splitting of a connect summand of M.

Our basic approach is to use the genus of an alternately sloped surface in the link exterior to bound the bridge distance. The foundations of this approach can be traced back to at least the solution to the cyclic surgery theorem [13], Gordon and Luecke's solution to the knot complement problem [20], and Rubinstein and Scharlemann's analysis of Heegaard splittings of manifolds with genus 2 Heegaard splittings [41]. We also make use of recent methods of Johnson [28], work of Campisi [11], and work of Tomova [48].

1.1. **Surgery and Essential Surfaces.** If the result of surgery on the link L in the 3-manifold M produces a 3-manifold with an essential surface S then S can be isotoped so that it is either disjoint from a regular neighborhood of L or intersects the link exterior $M(L)$ in an essential surface whose boundary is parallel to the surgery slope. We show in Section 3 that the genus of such a surface provides an upper bound for the bridge distance of L. It follows, for example, that cable knots have bridge distance at most 2 since they have an essential non-meridional annulus in the exterior. (This is also shown in [2].) The Cabling Conjecture of Gonzalez-Acuña and Short [29, Problem 1.79] states that if a knot in S^3 has a reducing surgery, then the knot is a cable knot and the surgery slope is the slope of the cabling annulus. The

conjecture has been shown to hold for many classes of knots including satellite knots [42], symmetric knots [14, 23, 33], persistently laminar knots [9, 10], alternating knots [34], many knots with essential tangle decompositions [24], knots where a surgery produces a connected sum of lens spaces [18], and knots that are band sums [45]. As a consequence of our main theorem, the cabling conjecture is verified for knots with bridge distance at least 3 with respect to a bridge sphere.

A characterization of hyperbolic knots in S^3 having toroidal surgeries is more elusive and examples are easily constructed by considering knots lying on knotted genus two surfaces in S^3. Additionally, many other examples of knots with toroidal surgeries are known (e.g. [15, 47]). It is known that the surgery slope of a toroidal surgery must be integral or half-integral [21]. The punctured torus in the knot exterior can be assumed to have no more than 2 boundary components [22]. Teragaito [47] has conjectured a relationship between the genus of the knot and the surgery slope; the conjecture has been proved for genus 2 knots [32]. All toroidal surgeries on Montesinos knots have been classified [50]. Our result gives the first universal restriction on hyperbolic knots having toroidal surgeries.

Furthermore, Corollary 3.2 shows that the genus of any non-meridonal essential surface in a link exterior can be used to give an upper bound on bridge distance independent of the number of boundary components. This result should be compared to results of Bachman and Schleimer [2] who show that euler characteristic of an essential surface in a knot exterior gives a bound on bridge distance.

1.2. **Surgery and Heegaard surfaces.** At least since Casson and Gordon's introduction [12] of the notion of strongly irreducible Heegaard surfaces, such surfaces have been almost as useful as essential surfaces in understanding 3-manifolds. Inspired by the notion of "thin position" introduced by Gabai [17] for knots in S^3 and by Scharlemann and Thompson [43] for handle decompositions of 3-manifolds, we show that the genus of a Heegaard surface in the surgered manifold can be used to bound the bridge distance of the surgery link.

The Berge conjecture (cf. [29, Problem 1.78]) states that every knot in S^3 with a lens space surgery is doubly primitive with respect to a genus two Heegaard surface for S^3. Recently, a number of research programs to prove the Berge conjecture have been proposed and partially completed. In [4], a two step program using knot Floer homology is outlined and the first step of the program is completed in [26]. In [49], the first step in a three step program is completed that would result in the proof of the Berge conjecture for tunnel number one knots. The

main result of this paper implies, since lens spaces have genus 1 Heegaard surfaces, that knots with a bridge sphere of distance greater than 3 do not admit lens space surgeries.

Small Seifert fibered spaces have Heegaard genus at most 2 [7]. Consequently, our work shows that knots in S^3 having surgeries producing small Seifert fibered spaces have bridge distance no larger than 4. Combining similar results with the Geometrization Theorem [36–38] shows that a hyperbolic knot in an aspherical, atoroidal closed orientable 3-manifold having bridge distance at least 7 with respect to a bridge surface does not have any non-hyperbolic surgeries.

Hyperbolic knots also have very few cosmetic surgeries [5, Theorem 1], i.e., surgeries along distinct slopes that produce the same 3-manifold. Indeed, it is conjectured [29, Problem 1.81] that hyperbolic knots have no so-called "exotic" cosmetic surgeries. The article [5] gives a thorough survey of known results on cosmetic surgery. We highlight just a few results.

Gabai [17] showed that no non-trivial knot in $S^2 \times S^1$ admits a non-trivial cosmetic surgery. Gordon and Luecke's solution to the knot complement conjecture shows that non-trivial knots in S^3 admit no cosmetic surgeries [20]. Lackenby has determined bounds on the denominator of the surgery coefficient of a cosmetic surgery [30].

In the present paper, we show that for knots with high bridge distance (relative to the Heegaard genus of the ambient manifold) the Heegaard genus strictly increases under non-trivial Dehn surgery, so such knots do not admit cosmetic surgeries.

Finally, we note that a number of authors have studied the effect of Dehn surgery on Heegaard surfaces. Moriah and Rubinstein [35] have shown that most surgeries on most knots preserve the isotopy classes of Heegaard splittings of their exteriors. (See also [16, 40].) Rieck [39] showed that most (but possibly not all) surgeries on an anannular knot strictly increase the Heegaard genus. Additionally, a similar philosophy to the one presented in the current paper is evident in the recent paper y Baker-Gordon-Luecke [3] that relates surgery distance, knot width, and the Heegaard genus of the surgered 3-manifold for certain knots in S^3.

2. Definitions

If X is an embedded submanifold of M, we let $\eta(X)$ denote an open regular neighborhood of X. The notation $|X|$ indicates the number of components of X.

A surface F properly embedded in a 3-manifold M is *essential* if it is incompressible, not boundary parallel, and not a 2–sphere bounding a 3-ball. We say F is *boundary-compressible* if there exists an embedded disk in M with interior disjoint from $F \cup \partial M$ and with boundary the endpoint union of an essential arc in F and an arc in ∂M. We will denote the genus of F by $g(F)$.

2.1. **Heegaard Splittings.** A *handlebody* is a compact 3-manifold homeomorphic to a closed regular neighborhood of a graph G properly embedded in $\mathbb{R}^3$. A *compression body* H is a connected 3-manifold homeomorphic to any component of a regular neighborhood in a 3-manifold N of $G \cup \partial N$. Here N may have empty boundary and G is an embedded graph, possibly with vertices in ∂N. The inclusion in H of ∂N is called the *negative boundary* of H and is denoted $\partial_- H$. We call $\partial H \setminus \partial_- H$ the *positive boundary* of H and denote it $\partial_+ H$. The inclusion into H of $G \cup \partial N$ is called a *spine* of H.

A *Heegaard splitting* for a 3-manifold M is a triple $(H, H_\downarrow, H_\uparrow)$ where H is a connected, closed, embedded, separating surface and $H_\downarrow$, $H_\uparrow$ are compression bodies with disjoint interiors such that $M = H_\downarrow \cup H_\uparrow$, $H = \partial_+ H_\downarrow = \partial_+ H_\uparrow$ and $\partial M = \partial_- H_\downarrow \cup \partial_- H_\uparrow$. The surface H is called a *Heegaard surface*.

2.2. **Bridge Surfaces.** The terminology presented in this section is an adaptation of that used by various authors including Hayashi-Shimokawa [25], Scharlemann-Tomova [44], and Taylor-Tomova [46]. Given a link L embedded in a manifold M we will denote by $M(L)$ the complement in M of an open regular neighborhood of L in M, i.e., $M(L) = M \setminus \eta(L)$.

We say that a Heegaard surface H for M transverse to L is a *bridge surface* for (M, L) if for each arc α of $L \cap H_\downarrow$ or $L \cap H_\uparrow$, there is an embedded disk $D \subset M$ with interior disjoint from L and H whose boundary consists of α and an open arc in $H \setminus L$. Any such disk D is called a *bridge disk* for L.

A *compressing disk* for H is an embedded disk in $M(L)$ whose interior is disjoint from H and whose boundary is essential in $H \setminus \eta(L)$. A bridge surface H is *weakly reducible* if there is a disjoint pair $D_\downarrow$, $D_\uparrow$, each a compressing disk or bridge disk, on opposite sides of H. If H has compressing disks or bridge disks on both sides but is not weakly reducible then we call H *strongly irreducible*.

Suppose that $D_\downarrow$ and $D_\uparrow$ are bridge disks on opposite sides of H such that $D_\downarrow \cap D_\uparrow$ is contained in L. If $D_\downarrow \cap D_\uparrow$ is a single point, then H is *perturbed*. If $D_\downarrow \cap D_\uparrow$ is two points, then H is *cancellable* and the component $K \subset \partial D_\downarrow \cup \partial D_\uparrow$ is called the *cancellable component* of L.

Note that a cancellable or perturbed bridge surface may not be weakly reducible.

We say that a component of L is *removable* if H is cancellable with canceling disks $D_\downarrow, D_\uparrow$ and there is a compressing disk disjoint from one of $D_\downarrow, D_\uparrow$ and intersecting the other in a single point in its boundary.

If H is perturbed, isotoping H across either $D_\uparrow$ or $D_\downarrow$ produces a new bridge surface for L intersecting L two fewer times. If H is cancellable, we can isotope H across $D_\downarrow \cup D_\uparrow$ to a surface H' containing K. The surface $R = H' \cap M(L)$ is called a *cancelled bridge surface* for (M, L). If H is removable, then there is a component $K \subset L$ such that H is isotopic to a bridge surface for $(M(K), L - K)$. See [44] for more details.

2.3. **Properly embedded surfaces in link complements.** If V is a union of tori then a *multislope* σ on V is an isotopy class in V of a union of essential curves, one in each component of V. If $V \subset \partial M$ and if $F \subset M$ is a properly embedded surface, we say that F defines a σ-slope on V if σ is represented by the union of components of $\partial F \cap V$. If σ and τ are multislopes on V, their *intersection number*, denoted $\Delta(\sigma, \tau)$, is calculated by taking the minimum intersection number (over all components of V) between minimally intersecting representatives of σ and τ in a single component of V. In particular, if $\Delta(\sigma, \tau) > 0$ then the the slopes are distinct in every component.

Lemma 2.1. *Suppose that H is a perturbed or cancellable but strongly irreducible bridge surface for a link $L \subset M$. Assume that L is not an unknot in 1-bridge position. Then H is cancellable. Furthermore, let K be a cancellable component of L that is not the unknot and let R be the cancelled bridge surface. Then all of the following hold:*

- *The slopes ρ and τ in $\partial\eta(K)$ defined by R and H, respectively, intersect in a single point.*
- *R is an essential surface in $M(L)$.*
- *Some component of R has genus strictly less than then genus of H.*
- *If $K = L$ and if M is closed, then after Dehn filling $\partial\eta(K) \subset \partial M(L)$ with slope ρ and capping $\partial R \cap \partial\eta(K)$ with disks, we obtain an incompressible surface $R(\rho)$ in the filled manifold $M(L)(\rho)$.*

Proof. Suppose, first, that H is perturbed with perturbing disks $D_\downarrow$ and $D_\uparrow$. By the claim from the proof of [44, Lemma 3.1], there is a complete collection of disjoint bridge disks and compressing disks for each side of H that contains the disks $D_\downarrow$ and $D_\uparrow$. Suppose that $L - H$

has more than one component or that $H-L$ is compressible on the same side of H as $D_\uparrow$. Then there would be a compressing disk or bridge disk D for H on the same side of H as $D_\uparrow$ that is entirely disjoint from $D_\downarrow$. If D is a bridge disk, the frontier D' of a regular neighborhood of D in $M-H$ is a compressing disk for $H-L$ that is disjoint from $D_\downarrow$. Similarly, the frontier of a regular neighborhood of $D_\downarrow$ in $M-H$ is a compressing disk for $H-L$ that is disjoint from from D'. Thus, H is weakly reducible, a contradiction. Hence, H is cancellable.

Suppose that $D_\downarrow$ and $D_\uparrow$ are bridge disks for the cancellable component $K \subset \partial D_\downarrow \cup \partial D_\uparrow$. We can obtain R by simultaneously boundary compressing $H \cap M(L)$ using the disks $D_\downarrow$ and $D_\uparrow$. Since these disks each intersect a component of $\partial(H \cap M(L))$ exactly once, each component of ∂R intersects a meridian of K exactly once.

Suppose that E is a compressing disk for $R \cap M(L)$. The isotopy of H to the surface H' containing K along the disks $D_\downarrow$ and $D_\uparrow$ moves E to a compressing disk for $H'-K$ in $M-K$ that is disjoint from $D_\downarrow \cup D_\uparrow$. Since one of $D_\downarrow$, $D_\uparrow$ is on the opposite side of H from E, this implies H is weakly reducible, a contradiction. Therefore R must be incompressible.

The loop K in H' is either separating or non-separating. If it is non-separating, then R is connected and has genus one less than the genus of H. If K is separating, then R has two components. Since K is not the unknot, neither of these components is a disk and, therefore, they both must have genus strictly less than the genus of H. Hence, the third claim.

Suppose that E is a boundary compressing disk for the incompressible surface R. Since ∂R lies in torus components of $\partial(M(L))$, this implies that R is a boundary parallel annulus. Hence, $L = K$ is a knot. Moreover, H is obtained from R by attaching the annulus $\eta(K) \cap H$ and so H is a torus and K is a torus knot. However, if K is not the unknot then R is boundary incompressible and, in particular, not boundary parallel.

Suppose that $K = L$ and that M is closed. Since K is not the unknot in 1-bridge position, K is an essential loop in H'. Since M is closed, H' is compressible to both sides in M. Hence, the Jaco handle addition theorem [27], applied to the 3-manifolds on either side of H', implies that $R(\rho)$ is incompressible in $M(L)(\rho)$. This is the fourth claim. $\square$

2.4. **Sloped Heegaard Surfaces.** Rather than working with a (3-manifold, link) pair (M, L), it is often advantageous to work entirely with the exterior of the link. To that end, we follow Campisi [11] and define the notion of a "sloped Heegaard splitting".

Let M be a compact, orientable manifold, possibly with boundary, containing a properly embedded link L. Let $N = M(L)$ and let $\partial_0 N = \partial N \setminus \partial M$ be a union of torus boundary components of N defined by L with σ a multislope on $\partial_0 N$.

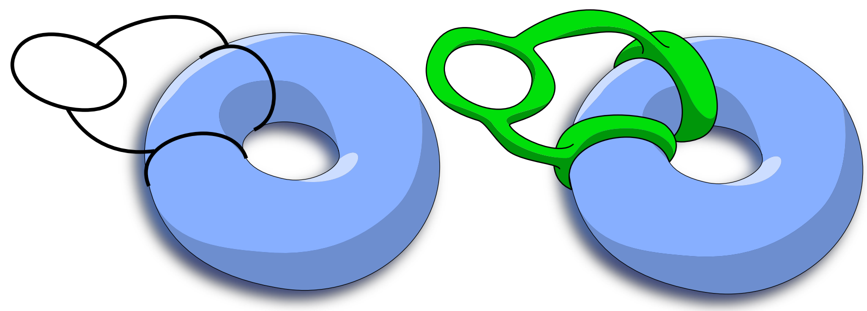

FIGURE 1. An spine and a boundary compression body.

In each component V_i of $\partial_0 N$, choose a collection of parallel simple closed curves α_i representing the slope $\sigma_i = \sigma \cap V_i$. Let Γ be a connected graph that is the union of the curves α_i with a properly embedded graph Γ^- in N having the property that any vertex of Γ^- contained in any V_i is also contained in some curve of some α_i. Let b_i be a regular neighborhood of a component of some α_i. Let $S_\downarrow$ be a regular neighborhood of Γ together with any component of ∂N that contains only vertices of Γ, as in Figure 1. We call $S_\downarrow$ a *boundary compression body* with slope σ. The union of Γ and any component of $\partial N \setminus \partial_0 N$ intersecting Γ only in vertices is called a *spine* for $S_\downarrow$. If the boundary compression body is either a 3–ball or the product of a component of ∂N with the closed interval, we call it *trivial*.

If the closure of the complement of $S_\downarrow \subset N$ is a second boundary compression body $S_\uparrow$ then we will say that $(S, S_\downarrow, S_\uparrow)$ is a *σ-sloped Heegaard splitting* for N, where $S = S_\downarrow \cap S_\uparrow = \partial_+ S_\downarrow = \partial_+ S_\uparrow$ is a properly embedded surface called a *σ-sloped Heegaard surface*. If the multislope σ is clear from the context, we drop it from the terminology and refer to *sloped Heegaard splittings* and *sloped Heegaard surfaces*. The following straightforward lemma explains our interest in sloped Heegaard splittings. Its proof is left to the reader.

Lemma 2.2. *If H is a bridge surface for (M, L) then $S = H \cap M(L)$ is a meridian sloped Heegaard surface for $N = M(L)$. Conversely, suppose that S is a σ-sloped Heegaard surface for N, and that $N(\sigma)$ is obtained by Dehn filling $\partial_0 N$ with slope σ. If L is the union of the*

cores of the filling tori then there is a bridge surface H for (M, L) such that $S = H \cap M(L)$.

A separating, properly embedded, possibly disconnected surface S in N is *weakly boundary reducible* if there are disjoint disks $D_\downarrow$, $D_\uparrow$, each either a compressing disk or a boundary compressing disk, on opposite sides of S. If S has compressing or boundary compressing disks on both sides but is not weakly boundary reducible then it is *strongly boundary irreducible*. Note that if H is a bridge surface for (M, L) that is perturbed or cancellable then the two bridge disks that intersect in one or two points in L define disjoint boundary compressing disks for $S = H \cap M(L)$. Thus, being weakly reducible is a more restrictive condition than being boundary weakly boundary reducible. However, S will be weakly boundary irreducible if and only if H is either weakly reducible, perturbed or cancellable.

Let σ be a multislope on $\partial_0 N$ and suppose that $V \subset \partial N \setminus \partial_0 N$ is a torus component. Let σ_V be a slope on V. A *σ_V-sloped bead* is a properly embedded annulus that is parallel into V and whose boundary has slope σ_V. Given a σ-sloped Heegaard surface S, we can form a $(\sigma \cup \sigma_V)$-sloped Heegaard surface S' by attaching an unknotted tube in $N \backslash S$ to a σ_V-sloped bead in V. We say that S' is *$(\sigma \cup \sigma_V)$-stabilized.* As recorded in the next lemma, "slope stabilization" is to sloped Heegaard splittings as "removable" is to bridge surfaces.

We allow the reader to work out the proof of the following:

Lemma 2.3. *If H is a bridge surface for (M, L) then H is removable if and only if $S = H \cap M(L)$ is σ-stabilized where σ is the slope of the meridian.*

2.5. **Sloped generalized Heegaard Splittings.** Campisi [11] defines a structure called a *sloped generalized Heegaard splitting* that generalizes sloped Heegaard surfaces in the same way that generalized Heegaard surfaces generalize Heegaard surfaces.

A *sloped generalized Heegaard splitting* [11] of N is a decomposition

$$N = ((S_1)_\downarrow \cup_{S_1} (S_1)_\uparrow) \cup_{F_1} ((S_2)_\downarrow \cup_{S_2} (S_2)_\uparrow) \cup_{F_2} \cdots \cup_{F_{m-1}} ((S_m)_\downarrow \cup_{S_m} (S_m)_\uparrow)$$

such that each of the $(S_i)_\uparrow$ and $(S_i)_\downarrow$ is a boundary compression body. The collection of surfaces $\cup_i F_i = \mathcal{F}$ is called the *thin surfaces* and the collection of surfaces $\cup_i S_i = \mathcal{S}$ is called the *thick surfaces* of the decomposition. We denote the sloped generalized Heegaard splitting by $(\mathcal{S}, \mathcal{F})$.

We note the following, which is implicit in [11]:

Lemma 2.4. *Every sloped generalized Heegaard splitting either has a non-empty collection of thin surfaces or consists of a single sloped Heegaard surface.*

If $(S, S_\uparrow, S_\downarrow)$ is a weakly reducible sloped Heegaard splitting then it is possible to *untelescope* S to obtain a sloped generalized Heegaard splitting of $M(L)$. Beginning with a handle and bead structure determined by $(S, S_\uparrow, S_\downarrow)$ one can rearrange the order of the handles and beads to obtain a sloped generalized Heegaard splitting for $M(L)$. If any thick level is weakly boundary reducible, the decomposition can be further untelescoped.

Theorems 3.4 and 3.7 in [11] imply the following:

Theorem 2.5. *Suppose the boundary of $M(L)$ is a collection of tori. Any σ-sloped Heegaard splitting $(S, S_\uparrow, S_\downarrow)$ of $M(L)$ can be untelescoped (possibly in zero steps) so that in the resulting sloped generalized Heegaard splitting $(\mathcal{S}, \mathcal{F})$*

(1) every component of $\mathcal{F}$ is incompressible,
(2) no component of $N - (\mathcal{S} \cup \mathcal{F})$ is a trivial cobordism between a component of $\mathcal{S}$ and a component of $\mathcal{F}$,
(3) and each component of $\mathcal{S}$ in the complement of $\mathcal{F}$ is either
  (a) slope stabilized, with slope a subset of σ
  (b) strongly irreducible, but weakly boundary reducible
  (c) strongly boundary irreducible.

For most of the results in this paper, it is also possible to use the results of Hayashi-Shimokawa [25], but Campisi's setup is more convenient for our purposes.

2.6. **Sweep-outs.** Let $(S, S_\uparrow, S_\downarrow)$ be a sloped Heegaard splitting for a manifold M. From the definition of a spine one can construct a map $\phi_S : M \to [0,1]$ such that $\phi_S^{-1}(0)$ is a spine for $S_\downarrow$, $\phi_S^{-1}(1)$ is a spine for $S_\uparrow$ and $\phi_S^{-1}(s) = S_s$ is properly isotopic to S for all $s \in (0,1)$. This function is called a *sweep-out* representing $(S, S_\uparrow, S_\downarrow)$.

More generally, if S_i is a thick surface with boundary in a sloped generalized Heegaard splitting that contains thin levels, then $(S_i, (S_i)_\uparrow, (S_i)_\downarrow)$ is a sloped Heegaard splitting of the submanifold $M_i = (S_i)_\downarrow \cup_{S_i} (S_i)_\uparrow$. As above one can construct a map $\phi_i : M_i \to [0,1]$ such that $\phi_i^{-1}(0)$ is a spine for $(S_i)_\downarrow$, $\phi_i^{-1}(1)$ is a spine for $(S_i)_\uparrow$ and $\phi_i^{-1}(s) = S_{i_s}$ is properly isotopic to the surface S_i for all $s \in (0,1)$. This function is called a *partial sweep-out* representing $(S_i, (S_i)_\uparrow, (S_i)_\downarrow)$.

2.7. **Distance.** The *curve complex* $\mathcal{C}(S)$ of a surface S with boundary is the simplicial complex whose vertices are isotopy classes of essential

(and not boundary parallel) simple closed curves in S and whose edges span pairs of isotopy classes of curves that have disjoint representatives. Higher dimensional simplices are defined by pairwise disjoint sets of curves. We make the vertex set of $\mathcal{C}(S)$ a metric space by defining the distance between two vertices as the number of edges in the shortest edge path between them.

Given a bridge surface H for a link $L \subset M$ or a sloped Heegaard surface S for $N = M(L)$, we define the curve complexes $\mathcal{C}(H)$ and $\mathcal{C}(S)$ to be the curve complexes for the surfaces $H \cap M(L)$ and S, respectively. The disk sets $\mathcal{H}_\uparrow$, $\mathcal{H}_\downarrow$ for H (respectively, S) are the sets of loops that bound compressing disks for $H \cap M(L)$ (S) in $H_\uparrow \cap M(L)$ ($S_\uparrow$) and $H_\downarrow \cap M(L)$ ($S_\downarrow$).

Definition 2.6. The *distance* $d_{\mathcal{C}}(H)$ is the distance in $\mathcal{C}(H)$ between $\mathcal{H}_\uparrow$ and $\mathcal{H}_\downarrow$.

As mentioned in the introduction, this is a non-trivial definition:

Theorem 2.7 ([6], Corollary 5.3)**.** *Given a set of surfaces $\{F_i\}$ and non-negative integers b, c, d, and g with $c \leq b$ such that if $g = 0$, then $b \geq 3$, and if $g = 1$, then $b \geq 1$, there exists an orientable 3-manifold M with $\partial M = \{F_i\}$ containing a c-component link L and a bridge surface H of genus g for (M, L) so that L is b-bridge with respect to H and $d_{\mathcal{C}}(H) \geq d$. Moreover H partitions the $\{F_i\}$ into any two prescribed sets as long as g is at least equal to the sum of the genera of the surfaces in each of the sets.*

The *arc and curve complex* $\mathcal{AC}(S)$ for a surface S with boundary is the simplicial complex whose vertices are isotopy classes of essential simple closed curves and essential properly embedded arcs. Edges span pairs of disjoint arcs/curves and higher dimensional simplices span larger sets of pairwise disjoint arcs/curves. As with the curve complex, we make its vertex set a metric space by declaring each edge of the curve complex to have length one and ignoring the higher dimensional simplices.

Given a sloped Heegaard surface S, the collection of boundary compressing disks and compressing disks for S intersect S in arcs and loops, respectively. These arcs and loops define vertices in the arc and curve complex of S. The *disk sets* of S are the sets $\mathcal{D}_\uparrow$, $\mathcal{D}_\downarrow$ of vertices of $\mathcal{AC}(S)$ defined by the intersection of all boundary compressing and compressing disks on the two sides of S.

Definition 2.8. The *distance* $d_{\mathcal{AC}}(S)$ of a sloped Heegaard surface S is the edge path distance in $\mathcal{AC}(S)$ from $\mathcal{D}_\uparrow$ to $\mathcal{D}_\downarrow$, i.e., the number of edges in the shortest edge path from a vertex of $\mathcal{D}_\uparrow$ to a vertex in $\mathcal{D}_\downarrow$.

The *bridge distance* of a bridge surface is simply the distance of the induced sloped Heegaard surface.

In general, given any two sets of arcs and curves $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{AC}(S)$, $d_{\mathcal{AC}(S)}(\mathcal{U}, \mathcal{V})$ is the length of the shortest path in $\mathcal{AC}(S)$ from a curve in $\mathcal{U}$ to a curve in $\mathcal{V}$.

Given an essential arc $\alpha \subset S$, let U be a closed regular neighborhood of α and the boundary component(s) of S containing the endpoints of α. If S is not an annulus or three-punctured sphere, then ∂U contains one or two essential loops disjoint from α. If S is a sloped Heegaard surface induced by a bridge surface H and α bounds a boundary compressing disk for S then a component of ∂U will bound a compressing disk for both H and S. Moreover, if α and β are disjoint essential arcs then there is an essential loop ℓ disjoint from both arcs, and thus from their regular neighborhoods. Thus the boundary loops of the regular neighborhoods defined by α and β are distance at most 2 in $\mathcal{C}(H)$. (They will not be disjoint if α and β have endpoints in the same boundary component of S.) Consequently:

Lemma 2.9. *If S is a sloped Heegaard surface induced by a bridge surface H such that H is not a 1-bridge or 2-bridge sphere then $d_{\mathcal{C}}(H) \leq 2d_{\mathcal{AC}}(S)$.*

This lemma, combined with Theorem 2.7, implies the existence of high distance sloped Heegaard splittings of arbitrary genus for links with an arbitrary number of components. Unless explicitly stated otherwise, from this point forward we will be measuring distance in terms of the arc and curve complex and will write $d(S) = d_{\mathcal{AC}}(S)$.

3. Essential surfaces bound distance

As before, let M be a compact, orientable manifold containing a properly embedded link L and let $N = M(L)$ and $\partial_0 N = \partial N \setminus \partial M$. Let σ and τ be multi-slopes on $\partial_0 N$ such that $\Delta(\sigma, \tau) > 0$. Let $S \subset N$ be a properly embedded surface with boundary slope σ and let T be a thick surface of a generalized sloped Heegaard splitting for N such that T has boundary slope τ. Assume that no component of S is a closed sphere.

Theorem 3.1. *Let ϕ be a (partial) sweepout of N with level surfaces $T_t = \phi^{-1}(t)$ such that ∂T_t intersects ∂S minimally for all $t \in (0,1)$. Assume the restriction $\phi|_S$ is Morse with critical points at distinct heights. If there are distinct regular values $a, b \in (0,1)$ of $\phi|_S$ such that for every $t \in (a,b)$, every arc in the intersection $T_t \cap S$ is essential in both*

surfaces then

$$d_{\mathcal{AC}(T)}(T_a \cap S, T_b \cap S) \leq \frac{-4\chi(S)}{|\partial S \cap \partial T|}$$

Proof. Every arc of $T_t \cap S$ has two (distinct) endpoints in $\partial T_t \cap \partial S$ so there are exactly half as many arcs as endpoints. Since $|\partial T_t \cap \partial S|$ is minimal for each t, the number of arcs in $T_t \cap S$ is constant for all regular values t and is equal to $|\partial T \cap \partial S|/2$.

Suppose that $c \in S$ is an index 1 critical point of $\phi|_S$ and that $v = \phi(c)$ is the associated critical value. Choose $\epsilon > 0$ sufficiently small so that $\phi^{-1}|_S[v-\epsilon, v+\epsilon]$ contains a unique critical point. As T_t passes through this critical point, arcs and circles of $T_{v-\epsilon} \cap S$ are banded together, by a single band, to obtain arcs and circles isotopic to the arcs and circles of $T_{v+\epsilon}$. If passing through the critical point involves banding a circle to itself or banding two circles together, then the isotopy classes of arcs of $T_{v+\epsilon} \cap S$ are the same as the isotopy classes of arcs of $T_{v-\epsilon} \cap S$. If passing through the critical point involves banding an arc to itself or banding an arc to a circle, then there is at most one isotopy class of an arc of $T_{v+\epsilon} \cap S$ that is not an isotopy class of an arc of $T_{v-\epsilon} \cap S$, and vice versa. Finally, if passing through the critical point involves banding two arcs together then there are at most two isotopy classes of arcs of $T_{v+\epsilon} \cap S$ that are not isotopy classes of arcs of $T_{v-\epsilon} \cap S$ and vice versa. If the isotopy classes of arcs of $T_{v-\epsilon} \cap S$ and $T_{v+\epsilon} \cap S$ differ, we say that c is an *active critical point*.

If $T_{v\pm\epsilon} \cap S$ contains an arc component α that is not isotopic to any component of $T_{v\mp\epsilon} \cap S$ we say that α is an *active arc* at c and v. This occurs at the saddles of S, as in Figure 2. The critical point c is *adjacent* to the active arc α. If $\alpha \subset T_{v-\epsilon} \cap S$ then α is a *pre-active arc* and if $\alpha \subset T_{v+\epsilon} \cap$ then α is a *post-active arc*. Since the number of arcs of $T_t \cap S$ is constant for all regular values t, the numbers of pre-active and post-active arcs at each critical value are equal.

Let $\mathcal{C}$ be the set of all critical points of $\phi|_S$ adjacent to some active arc and let $\mathcal{V}$ be the image of $\mathcal{C}$ under $\phi|_S$. By the above argument, the number of post-active arcs is at most $2|\mathcal{V}|$.

Suppose that α is a pre-active arc at a critical point c. The post-active arcs at c are obtained by either banding α to another pre-active arc β at c, in which case we say that α is *paired*, or by banding α to itself or to a circle component of $T_{v-\epsilon} \cap S$, in which case we say that α is *solitary*.

Suppose α is an active arc at c. If α is banded at c to a circle γ, or if α is banded at c to itself so that a circle γ is created, then, by the

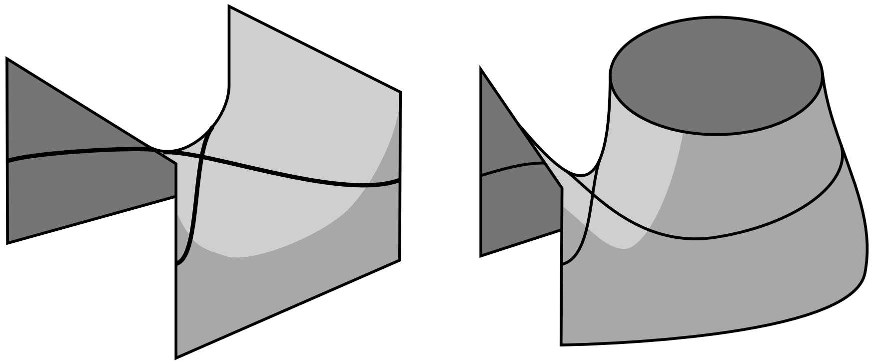

FIGURE 2. Active arcs occur before and after saddles.

definition of active arc, γ is essential in S. In either case, we say that γ is an *active circle*.

Let A be the union of all active arcs and circles for all critical points of $\phi|_S$. Let P be the closure of $S \setminus A$ and denote its components by $P_1, \ldots, P_r$. The boundary of each P_k is the union of arcs or circles lying in ∂N and arcs and circles that lie in A and each P_k contains at most one active critical point. Let b_k be the number of active arcs in ∂P_k and define the *index* of P_k to be:

$$J(P_k) = b_k/2 - \chi(P_k).$$

Notice that $-\chi(S) = \sum_k J(P_k)$ since each active arc shows up twice in ∂P and since euler characteristic increases by one when cutting along an arc.

Fix k. By hypothesis, $P_k \neq S^2$. If P_k is a disk, its boundary cannot be an active circle or contain only a single active arc as active circles and arcs are essential in S. Thus, if P_k is a disk, $J(P_k) \geq 0$. If P_k is not a disk, $-\chi(P_k) \geq 0$. Thus, if P_k does not contain an active critical point then its index is non-negative.

Otherwise, assume $\phi|_{P_k}$ does contain an active critical point $c \in P_k$ and let α be a pre-active arc at c. If α is a paired arc at c, then $b_k \geq 4$ since there must be at least two pre-active arcs and two post active arcs. Then $J(P_k) \geq (4/2) - 1 = 1$.

If α is a solitary pre-active arc at $c \in \mathcal{C} \cap P_k$, then let γ be the circle that is either banded to α at c or that results from banding α to itself at c. In this case, γ is essential so P_k is not a disk and there are two active arcs in the boundary of P_k so $J(P_k) \geq (2/2) - 0 = 1$.

We conclude that $J(P_k)$ is non-negative if P_k contains no active critical points and $J(P_k) \geq 1$ if P_k contains an active critical point. Thus, $|\mathcal{V}| \leq -\chi(S)$.

Let $v_1 < v_2 < \ldots < v_r$ be the elements of $\mathcal{V}$. Label the arc components of $T_a \cap S$ by $\alpha_1^0, ..., \alpha_n^0$. The arcs of $T_{v_1-\epsilon} \cap S$ are isotopic to the arcs $T_a \cap S$. Give each arc of $T_{v_1-\epsilon} \cap S$ the same label as the isotopic arc in $T_a \cap S$. For each pre-active arc α_i^0 at v_1, there is a post-active arc α_i^1 such that some endpoint of α_i^1 is not separated in ∂S from some endpoint of α_i^0. There may not be a unique such arc, but since the number of pre-active arcs equals the number of post-active arcs and since the post active arcs are obtained (up to proper isotopy) by attaching bands to the pre-active arcs and circles, we may choose distinct post-active arcs α_i^1 for each pre-active arc α_i^0, as in Figure 3. If an arc of $T_{v_1+\epsilon} \cap S$ is not isotopic to a post-active arc at v_1, it is isotopic to an arc of $T_a \cap S$ and we give it the same label as that arc. Thus, an arc of $T_{v_1+\epsilon} \cap S$ that is not post-active at v_1 is labelled α_i^0 for some i, while an arc that is a post-active arc at v_1 is labelled α_i^1 for some $i \in \{1, \ldots, n\}$.

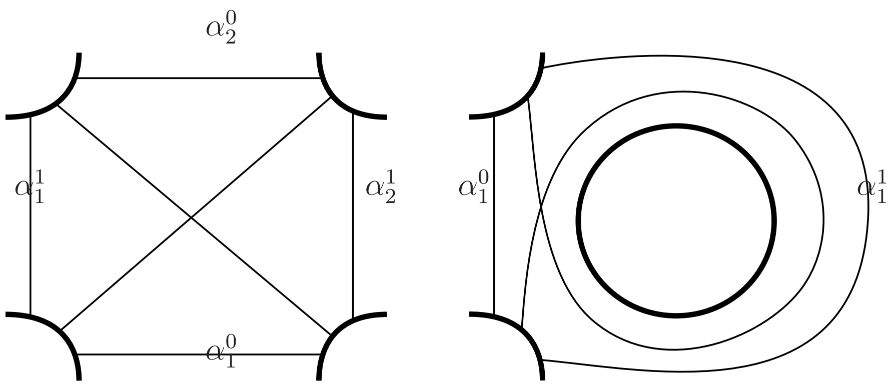

FIGURE 3. The naming scheme for arcs.

Assume that for a critical value v_j and for each arc of $T_{v_j-\epsilon} \cap S$ we have a label of the form α_i^k where $k \in \{0, \ldots, j-1\}$ and $i \in \{1, \ldots, n\}$. If an arc $\alpha \subset T_{v_j+\epsilon} \cap S$ is isotopic to an arc of $T_{v_j-\epsilon} \cap S$, give it the same label as that arc. Otherwise, it is a post-active arc at v_j and is obtained by attaching a band to some pre-active arc with label α_i^k. Give α the label α_i^{k+1}. Once again, there is not a unique choice of labels on the arcs of $T_{v_j+\epsilon} \cap S$, but there is a consistent choice of labels, and that is all we need. The arc components of $T_{v_{j+1}-\epsilon} \cap S$ are isotopic to the arc components of $T_{v_j+\epsilon} \cap S$ and we give them the corresponding labels. Consequently, by induction, for all $v_k \in \mathcal{V}$, we have labels on all the

arcs of $T_{v_k \pm \epsilon} \cap S$. The arcs of $T_b \cap S$ are isotopic to the arcs of $T_{v_r+\epsilon} \cap S$ and we give them the same labels.

Let q_i be such that $\alpha_i^{q_i}$ is a label of an arc of $T_b \cap S$. Identify each T_t with T so that for fixed i, the sequence $\alpha_i = (\alpha_i^0, \ldots, \alpha_i^{q_i})$ is a sequence of points in $\mathcal{AC}(T)$. If v_k is a critical value, then (under the identification of T_t with T) the pre-active arcs at a point $v_k \in \mathcal{V}$ are disjoint from the post-active arcs at v_k and so they are distance 1 in $\mathcal{AC}(T)$. Hence,

$$d_{\mathcal{AC}(T)}(T_a \cap S, T_b \cap S) \leq \min\{q_1, \ldots, q_n\} \leq \frac{1}{n}(q_1 + \ldots + q_n).$$

The number $q_1 + \ldots + q_n$ is equal to the total number of post-active arcs. Since each component P_k containing an active critical point contains at most two post-active arcs, we have:

$$q_1 + \ldots + q_n \leq 2|\mathcal{V}| \leq -2\chi(S).$$

As noted above, the number n of arcs is half of $|\partial S \cap \partial T|$, so we have:

$$d_{\mathcal{AC}(T)}(T_a \cap S, T_b \cap S) \leq \frac{-2\chi(S)}{n} = \frac{-4\chi(S)}{|\partial S \cap \partial T|}.$$

□

Corollary 3.2. *Let N, T and S be defined as above and assume that S is essential. Then*

$$d(T) \leq 2 + \frac{8g(S) - 8|S| + 4|\partial S|}{|\partial S \cap \partial T|}$$

Proof. Let $\phi\colon N \to [0,1]$ be a (partial) sweepout of N such that $T_t = \phi^{-1}(t)$ is isotopic to T for all $t \in (0,1)$. We can isotope S so that, for all $t \in (0,1)$, ∂T_t and ∂S are transverse and intersect minimally and so that $\phi|_S$ is Morse with critical points at distinct heights. Note that since S is essential and $\partial_0 N$ is a collection of tori, all arcs of $T_t \cap S$ are essential in T.

Let $\mathcal{D}_\uparrow$ and $\mathcal{D}_\downarrow$ be the disk sets in $\mathcal{AC}(T)$ of T corresponding to the τ-sloped boundary compression bodies on either side of T. Assume that $d(T) = d_{\mathcal{AC}(T)}(\mathcal{D}_\uparrow, \mathcal{D}_\downarrow) \geq 2$. For small ϵ, any arc in $(T_\epsilon \cap S)$ lies in $\mathcal{D}_\downarrow$ and any arc of $(T_{1-\epsilon} \cap S)$ lies in $\mathcal{D}_\uparrow$.

If there exists a regular value $t \in (\mathcal{D}_\downarrow, \mathcal{D}_\uparrow)$ with arcs α, β of $T_t \cap S$ such that $\alpha \in \mathcal{D}_\downarrow$ and $\beta \in \mathcal{D}_\uparrow$ then $d(T) \leq 1$. If there exists a critical value $v \in (0,1)$ such that there exist arcs $\alpha \subset T_{v-\epsilon} \cap S$ and $\beta \subset T_{v+\epsilon} \cap S$ such that α is in $\mathcal{D}_\downarrow$ or $\mathcal{D}_\uparrow$ and β is in $\mathcal{D}_\uparrow$ or $\mathcal{D}_\downarrow$ respectively, then once again $d(T) \leq 1$.

Otherwise, let $0 = w_0, w_1, w_2, \ldots, w_n = 1$ be the critical values of $\phi|_S$ and let $I_i = [w_i, w_{i+1}]$. The previous paragraph shows that there exist

$i, k \in \mathbb{N} \cup \{0\}$ so that the interval $[u, v] = I_i \cup I_{i+1} \cup \ldots \cup I_{i+k}$ has the following properties:

- For all regular values $t \in (u, v)$, all arcs of $T_t \cap S$ are essential in S
- If t is a regular value between u and the next smallest critical value of $\phi|_S$, there exists an arc of $T_t \cap S$ that is in $\mathcal{D}_\downarrow$.
- If t is a regular value between v and the next biggest critical value of $\phi|_S$, there exists an arc of $T_t \cap S$ that is in $\mathcal{D}_\uparrow$.

Letting $a = u + \epsilon$ and $b = v - \epsilon$ for very small ϵ,

$$d(T) = d_{\mathcal{AC}(T)}(\mathcal{D}_\downarrow, \mathcal{D}_\uparrow) \leq 2 + d_{\mathcal{AC}(T)}(T_a \cap S, T_b \cap S).$$

Hence, by Theorem 3.1,

$$d(T) \leq 2 + \frac{-4\chi(S)}{|\partial S \cap \partial T|}.$$

The Euler characteristic of S can be calculated from its genus and boundary as

$$-\chi(S) = 2g(S) - 2|S| + |\partial S|,$$

so we have the desired bound. □

3.1. **Other surfaces bound distance.** The following result is an adaptation of Theorem 5.1 of [2] and a generalization of Proposition 4.3 of [48]. It is needed to fill the gap left by Corollary 3.2 when comparing surfaces with the same boundary slope. This result requires additional definitions: A surface S in $M(L)$ is *c-essential* if it is essential, boundary incompressible, and there is no essential curve δ in S that bounds a once punctured disk in $M(L)$. If there is such a curve, the disk it bounds is a *cut-disk* and the surface is said to be *cut-compressible*. We use the terminology *c-disk* to denote a compressing disk or a cut-disk.

Theorem 3.3. *Let L be a link embedded in a compact oriented manifold M. Suppose T is a bridge surface for L and suppose F is a c-essential, possibly punctured surface that is not an annulus. Then $d_{\mathcal{C}}(T) \leq max(3, 2g(F) + |F \cap L|)$. Furthermore, if F is a closed surface, $d_{\mathcal{C}}(T) \leq max(1, 2g(F))$.*

Proof. Consider a sweet-out ϕ of M induced by the bridge surface T. By standard arguments, we may assume that $\phi|_F$ is Morse with critical points at distinct heights and, in particular, each surface $\phi^{-1}(t)$ for fixed t intersects F in at most one figure-eight curve or in at most one curve that also contains a point of $S \cap L$.

Let $T_\uparrow$ and $T_\downarrow$ be the compression bodies T bounds in M. As in the proof of Corollary 3.2, for small ϵ, every curve of $F \cap \phi^{-1}(\epsilon)$ bounds

a disk in F that is a compressing disk for $T_\downarrow$ and every curve of $F \cap \phi^{-1}(1-\epsilon)$ bounds a disk in F that is a compressing disk for $T_\uparrow$. Let $[t_0, t_1]$ be the smallest interval so that a curve in $F \cap \phi^{-1}(t_0 - \epsilon)$ bounds a possibly punctured disk in F that is a c-disk for $\phi^{-1}(t_0 - \epsilon)$ contained in $T_\downarrow$ and a curve in $F \cap \phi^{-1}(t_1 + \epsilon)$ bounds a possibly punctured disk in F that is a c-disk for $\phi^{-1}(t_0 + \epsilon)$ contained in $T_\uparrow$. It is clear that if such a t_0, t_1 pair doesn't exist or if $t_0 = t_1$, then there are c-disks on opposite sides of T with disjoint boundaries. By Proposition 4.1 of [48], which shows that every cut-disk is distance at most one from some compressing disk on the same side, it follows that $d(T) \leq 3$. In the special case when F is closed, we know that both c-disks are compressing disks and $d(T) \leq 1$.

Suppose that $t_0 < t_1$. Let $c_0, c_1, c_2, .., c_{n-1}$ be all critical values in the interval $[t_0, t_1]$ so $c_1 = t_0$ and $c_n = t_1$. Let r_i be a regular value so that $c_i < r_i < c_{i+1}$ and let $r_0 = c_0 - \epsilon$ and $r_n = c_{n-1} + \epsilon$. Note that any curve in $F \cap \phi^{-1}(r_i)$ for $1 \leq i \leq n$ that is inessential in S is also inessential in T by the minimality of $t_1 - t_0$. Also, note that by Lemma 2.9 of [48], $F \cap \phi^{-1}(r_i)$ for $0 \leq i \leq n$ always contains curves essential in F as otherwise F can be isotoped into a compression body.

Cut F along the collection of essential in F curves of $(F \cap \phi^{-1}(r_0)) \cup ... \cup (F \cap \phi^{-1}(r_n))$. Consider first the set of all components that don't lie entirely above $\phi^{-1}(r_n)$ or entirely below $\phi^{-1}(r_0)$. Note that by definition none of these components are disks or punctured disks. Each component has boundary on at most two levels $\phi^{-1}(r_i)$ and $\phi^{-1}(r_{i+1})$ and thus can be associated with a unique critical point c_i. Furthermore, at least one of the components associated with c_i must have boundary in both $\phi^{-1}(r_i)$ and $\phi^{-1}(r_{i+1})$ as otherwise F could be isotoped to be disjoint from T, a contradiction. Let $\{P_i\}$ be the set of all components associated with c_i that have boundary in both $\phi^{-1}(r_i)$ and $\phi^{-1}(r_{i+1})$.

Let $\gamma_0, \dots \gamma_n$ be a path in the curve complex from a curve in $F \cap \phi^{-1}(r_0)$ that bounds a c-disk for T in $T_\downarrow$ to a curve in $F \cap \phi^{-1}(r_n)$ that bounds a c-disk for T in $T_\uparrow$ so that γ_i is a curve in $F \cap \phi^{-1}(r_i)$. We can do this since, for each i, $F \cap \phi^{-1}(r_i)$ contains a curve that is essential in F and, by standard arguments, $d(\gamma_i, \gamma_{i+1}) \leq 1$ for all i.

Claim: If each of $\{P_i\}, \{P_{i+1}\}, ..., \{P_j\}$ consists entirely of annuli, then, after possibly rechoosing γ_j, $d(\gamma_i, \gamma_j) = 0$

*P*roof: As $\{P_k\}$ consists entirely of annuli for $i \leq k \leq j$, then each component of $\cup_{k=i}^{j} \{P_i\}$ is an annulus. Hence, for any essential curve γ_i in $F \cap \phi^{-1}(r_i)$, there is an essential curve γ'_j in $F \cap \phi^{-1}(r_j)$ so that $d(\gamma_i, \gamma'_j) = 0$, namely the other boundary component of the annulus in $\cup_{k=i}^{j} \{P_i\}$ bounded by γ_i. As the distance between any two curves in

$F \cap \phi^{-1}(r_j)$ and $F \cap \phi^{-1}(r_{j+1})$ is at most one, we can replace γ_j by γ'_j in the path $\gamma_0, ..., \gamma_n$. □

Let l be the number of indices $1 \leq i \leq n$ for which $\{P_i\}$ does not consist entirely of annuli. By the Claim, $d(\gamma_0, \gamma_n) \leq l$. Furthermore, each $\{P_i\}$ that does not consist entirely of annuli has a total Euler characteristic at most -1. The Euler characteristic of F is equal to the sum of the Euler characteristics of the $\{P_i\}$ and the Euler characteristic of the components of F above $\phi^{-1}(r_n)$ and those below $\phi^{-1}(r_0)$. Note that each of the later components can have at most one disk so $\chi(F) \leq -l + \delta$ where δ is the total number of disk components so $\delta = 0, 1$ or 2.

Let $\mathcal{D}_\uparrow$ and $\mathcal{D}_\downarrow$ be the collection of all curves in the curve complex of the punctured surface T that bound compressing disks for T in the exterior of L above and below T respectively. Then

$$\begin{aligned} d(T) &\leq d(\gamma_0, D_\downarrow) + d(\gamma_0, \gamma_n) + d(\gamma_n, D_\uparrow) \\ &\leq d(\gamma_0, D_\downarrow) + l + d(\gamma_n, D_\uparrow) \\ &\leq d(\gamma_0, D_\downarrow) - \chi(F) + \delta + d(\gamma_n, D_\uparrow). \end{aligned}$$

If the components of F above $\phi^{-1}(r_n)$ contain a disk component then that contributes to the value of δ but $d(\gamma_n, D_\uparrow) = 0$, otherwise $d(\gamma_n, D_\uparrow) = 1$. Similarly for the components of F below $\phi^{-1}(r_0)$. Thus $d(T) \leq 2 - \chi(F) = 2g(F) + |L \cap F|$.

□

Tomova demonstrated a bound for the distance of T if S is a Heegaard surface for $M(L)$. We use this result to handle one case of Theorem 5.1.

Theorem 3.4 (Theorem 10.3 of [48]). *Suppose that L is a non-trivial knot in a closed, irreducible, orientable 3-manifold M. Let T be a bridge surface for (M, L). If T is a sphere, assume that $|T \cap L| \geq 6$. If S is a Heegaard surface for $M(L)$, then*

$$d_{\mathcal{C}}(T) \leq 2g(S).$$

4. Double Sweep-outs and the Graphic

As before, let N be a compact, orientable 3-manifold with incompressible boundary and let $\partial_0 N$ be a union of torus components of ∂N. Let T be a sloped Heegaard surface for N inducing a multislope τ on a subset of $\partial_0 N$. Let S be either another sloped Heegaard surface for N or a thick surface of a sloped generalized Heegaard splitting of N. Assume that S induces a multi-slope σ on a subset of $\partial_0 N$ and assume that $\Delta(\sigma, \tau) > 0$.

Let ϕ_T be a sweep-out of N associated to T and let ϕ_S be either a sweep-out or a partial sweep-out of N associated to S. Assume that the boundary of each level surface (possibly disconnected) of ϕ_S intersects the boundary of each level surface of ϕ_T minimally in each V_i. (This is possible, for example, by choosing a flat metric on each V_i and requiring that the boundaries of the level sets be geodesics.)

Consider the product map $\phi_T \times \phi_S : N \to [0,1] \times [0,1]$. Each point (t,s) in the square represents a pair of surfaces $T_t = \phi_T^{-1}(t)$ isotopic to the punctured surface T and $S_s = \phi_S^{-1}(s)$ isotopic to the punctured collection of surfaces S. The *graphic* is the subset of the square consisting of all points (t,s) where T_t and S_s are tangent. We say that $\phi_T \times \phi_S$ is *generic* if it is stable [28] on the complement of the spines and each arc $\{t\} \times [0,1]$ and $[0,1] \times \{s\}$ contain at most one vertex of the graphic. The vertices in the interior of the graphic are valence four (crossings) and valence two (cusps). By general position of the spines, the graphic is incident to the boundary of the square in only a finite number of points. The vertices in the corners of the boundary correspond to points of intersection in $\partial_0 N$ between a spine of ϕ_S and a spine of ϕ_T. These vertices will have valence $|\partial T \cap \partial S|/4$. All other vertices in the boundary are valence one or two. We model our analysis of the graphic on that of Johnson [28].

For $(t,s) \in (0,1) \times (0,1)$ a regular value of $\phi_T \times \phi_S$ we say that T_t is *essentially above* S_s if there exists a component $l \subset S_s \cap T_t$ that is an essential arc or circle in S_s but bounds a compressing or boundary compressing disk for S_s contained in $(S_s)_\downarrow$. Similarly, we say that T_t is *essentially below* S_s if there exists a component $l \subset S_s \cap T_t$ that is an essential arc or circle in S_s and bounds a compressing or boundary compressing disk for S_s contained in $(S_s)_\uparrow$.

Let Q_a (Q_b) denote the points in $(0,1) \times (0,1)$ such that T_t is essentially above (below) S_s.

Remark 4.1. Note that if ϕ_S is a sweep-out (rather than a partial sweep-out), then a neighborhood of the bottom of the graphic is contained in Q_a and a neighborhood of the top of the graphic is contained in Q_b, for if ϵ is small, the components of $T_t \cap S_\epsilon$ are essential in S_ϵ and inessential in T_t for any $t \in (0,1)$.

If ϕ_S is a partial sweep-out, then any level surface for ϕ_S is isotopic to a collection of thick surfaces for a sloped generalized Heegaard splitting for N. Hence, $\phi_s^{-1}([0,s_0]) = (S_{s_0})_\downarrow$ and $\phi_s^{-1}([s_0,1]) = (S_{s_0})_\uparrow$ are boundary compression bodies. Let e be an edge of a spine of $\phi_s^{-1}([0,s_0])$. Since we have assumed ϕ_T is a full sweep-out of M and $\phi_T \times \phi_S$ is

generic, there exists a level surface, $\phi_t^{-1}(t_0)$, that intersects e transversely. Hence, for ϵ sufficiently small, (t_0, ϵ) is contained in Q_a. Similarly, there exists t_1 such that for ϵ sufficiently small, $(t_1, 1-\epsilon)$ is contained in Q_b.

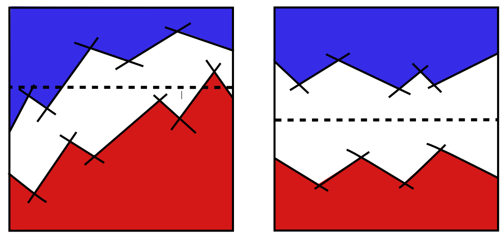

FIGURE 4. Spanning and splitting graphics.

4.1. **Spanning.**

Definition 4.2. The sweep-out or partial sweep-out ϕ_S *essentially spans* the sweep-out ϕ_T if there exist s_0, t_1, t_2 in $(0,1)$ such that $(t_1, s_0) \in Q_a$ and $(t_2, s_0) \in Q_b$, as on the left in Figure 4. Observe that if $(Q_a \cap Q_b)$ is not empty, then ϕ_S spans ϕ_T since Q_a and Q_b are open subsets of $(0,1) \times (0,1)$. Additionally, since Q_a and Q_b are open, we can assume $t_1 < t_2$.

Theorem 4.3. *Suppose that ϕ_S is a sweep-out or partial sweep-out that essentially spans the sweep-out ϕ_T and that S is not an annulus. Then S is weakly boundary reducible.*

Proof. Choose s_0, t_1, t_2 as in Definition 4.2. Without loss of generality, $t_1 < t_2$. Let $\widehat{T} = T_{t_1} \cup T_{t_2}$. Let $l_k \subset S_{s_0} \cap T_{t_k}$ be a component such that l_k is essential in S_{s_0} and bounds a compressing or boundary compressing disk D_k for S_{s_0} lying in $(S_{s_0})_\downarrow$ if $k = 1$ and in $(S_{s_0})_\uparrow$ if $k = 2$. Since T_{t_1} and T_{t_2} are disjoint, ∂D_1 and ∂D_2 are disjoint. $\square$

4.2. **Splitting.**

Definition 4.4. A sweep-out or partial sweep-out ϕ_S *essentially splits* a sweep-out ϕ_T for N if there exists $s_0 \in (0,1)$ such that for all $t \in (0,1)$, the point $(t, s_0) \notin (Q_a \cup Q_b)$, as on the right in Figure 4.

Although we will only apply the following lemma to sweep-outs, we have included a more general proof that suffices for both sweep-outs and partial sweep-outs.

Lemma 4.5. *Suppose ϕ_S is a generic sweep-out or a partial sweep-out that essentially splits the sweep-out ϕ_T. If ϕ_S is a partial sweep-out, assume the surface portions F_0 and F_1 of the spines $\phi_S^{-1}(0)$ and $\phi_S^{-1}(1)$ respectively are incompressible and boundary incompressible. Assume also that $d(T) \geq 3$.*

Then, for each s such that $[0,1] \times \{s\}$ is disjoint from $Q_a \cup Q_b$ in $(0,1) \times (0,1)$, there exist values $u_s < v_s \in [0,1]$ such that for sufficiently small $\epsilon > 0$, the following hold:

(1) For every $t \in (u_s, v_s)$, every arc in $T_t \cap S_s$ is essential in both S_s and T_t.
(2) Some arc or loop of $T_{u_s - \epsilon} \cap S_s$ bounds a compressing disk or boundary compressing disk on the negative side of $T_{u_s-\epsilon}$.
(3) Some arc or loop of $T_{v_s+\epsilon} \cap S_s$ bounds a compressing disk or boundary compressing disk on the positive side of $T_{v_s+\epsilon}$.

Proof. Fix $s_0 \in (0,1)$ satisfying the above hypotheses. By assumption, each point (t, s_0) is disjoint from Q_a and Q_b for $t \in [0,1]$. Since $d(T) \neq 0$, N is irreducible. We will begin by showing that every arc or loop in the intersection $S_{s_0} \cap T_t$ that is trivial in T_t must also be trivial in S_{s_0}.

If there were a trivial loop or arc in T_t that was essential in S_{s_0}, then an innermost such loop or outermost such arc would bound or cobound a disk D in T_t. Isotope D fixing $\partial D \cap S_{s_0}$ so that the number of components of $D \cap (F_0 \cup F_1 \cup S_{s_0})$ is minimized. Suppose α is an outermost arc or innermost loop of $D \cap (F_0 \cup F_1 \cup S_{s_0})$ in D. If α is contained in $F_0 \cup F_1$, then, by irreducibility of N and incompressibility and boundary incompressibility of $F_0 \cup F_1$, we can isotope D to decrease the number of components of $D \cap (F_0 \cup F_1 \cup S_{s_0})$. If α is contained in S_{s_0}, then it must be trivial in S_{s_0}. Since α is trivial in both S_{s_0} and D and N is irreducible, there is an isotopy of D that decreases the number of components of $D \cap (F_0 \cup F_1 \cup S_{s_0})$. In either case, we arrive at a contradiction, so we can assume the interior of D is disjoint from $F_0 \cup F_1 \cup S_{s_0}$. Hence, after an isotopy, the disk D is completely contained in either $(S_{s_0})_\downarrow$ or $(S_{s_0})_\uparrow$, contradicting the assumption that (t, s_0) is disjoint from Q_a and Q_b. We conclude that every arc or loop of $S_{s_0} \cap T_t$ is either essential in S_{s_0} or trivial in both surfaces.

On the other hand, a loop or arc in the intersection may be trivial in S_{s_0} but essential in T_t. In fact, for values of t near 0, such loops/arc must exist and will bound compressing disks and boundary compressing disks, respectively, in $(T_t)_\downarrow$. For t near 1, such loops will bound disks in $(T_t)_\uparrow$. Let $P_a \subset (0,1)$ be the set of all values of t such that $T_t \cap S_{s_0}$ contains a component that is essential in T_t and bounds a compressing disk or boundary compressing disk in $(T_t)_\downarrow$. Let $P_b \subset (0,1)$ be the

set of all values of t such that $T_t \cap S_{s_0}$ contains a component that is essential in T_t and bounds a compressing disk or boundary compressing disk in $(T_t)_\uparrow$. Note that P_a and P_b are open subsets of $(0, 1)$.

Claim: If $P_a \cap P_b \neq \emptyset$ or if the closure of $P_a \cup P_b$ in $(0, 1)$ is all of $(0, 1)$, then $d(T) \leq 2$.

Proof of claim: If $P_a \cap P_b \neq \emptyset$, then $d(T) \leq 1$ by definition. Suppose then that the closure of $P_a \cup P_b$ in $(0, 1)$ is all of $(0, 1)$. Then there is some value $t_0 \in (0, 1)$ such that, for all ϵ sufficiently small, $t_0 - \epsilon \in P_a$ and $t_0 + \epsilon \in P_b$. There are two cases to consider: (t_0, s_0) is contained in the interior of an edge of the graphic and (t_0, s_0) is contained in a vertex of the graphic.

If (t_0, s_0) is contained in an edge of the graphic, then it corresponds to a saddle point or a central singularity of ϕ_T restricted to S_{s_0}. In the saddle case, the arc or loop of $T_{t_0-\epsilon} \cap S_{s_0}$ bounding a compressing or boundary compressing disk below $T_{t_0-\epsilon}$ is at most distance one in the arc and curve complex of T from the arc or loop of $T_{t_0+\epsilon} \cap S_{s_0}$ bounding a compressing or boundary compressing disk above $T_{t_0+\epsilon}$. In the central singularity case, the collection of isotopy classes of arcs in $T_{t_0-\epsilon} \cap S_{s_0}$ is equal to the collection of isotopy classes of arcs in $T_{t_0+\epsilon} \cap S_{s_0}$. In either case, we conclude $d(T) \leq 1$.

If (t_0, s_0) is a vertex of the graphic, then it corresponds to a cusp or a valence 4 vertex. In the case of a cusp, $(t_0 - \epsilon, s_0)$ and $(t_0 + \epsilon, s_0)$ are contained in the same component of the complement of the graphic. Hence, the collection of isotopy classes of arcs in $T_{t_0} \cap S_{s_0}$ is equal to the collection of isotopy classes of arcs in $T_{t_0+\epsilon} \cap S_{s_0}$ and $d(T) \leq 1$. In the case of a vertex, there is a path in $(0, 1) \times (0, 1)$ with endpoints $(t_0 - \epsilon, s_0)$ and $(t_0 + \epsilon, s_0)$ that meets the graphic in two edge singularities. Hence, $T_{t_0-\epsilon} \cap S_{s_0}$ is obtained from $T_{t_0+\epsilon} \cap S_{s_0}$ by passing through two saddles and $d(T) \leq 2$. To visualize this rotate Figure 5 by 90 degrees. □(Claim)

By the claim, using the hypothesis that $d(T) \geq 3$, we can assume that the closure of $P_a \cup P_b$ in $(0, 1)$ is not all of $(0, 1)$.

Since P_a contains all values near 0, P_b contains all values near 1 and the closure of $P_a \cup P_b$ is not all of $(0, 1)$, there is some closed interval component $[u, v]$ in the complement of $P_a \cup P_b$ with the property that $u - \epsilon \in P_a$ and $v + \epsilon \in P_b$. For every $t \in (u, v)$, each arc of $T_t \cap S_{s_0}$ is either essential in both surfaces or trivial in both surfaces. However, as the intersection of ∂S_{s_0} and ∂T_t is minimal, no arc of $T_t \cap S_{s_0}$ can be trivial in both surfaces. Therefore, if $t \in (u, v)$ all the arcs of $T_t \cap S_{s_0}$ are essential in both surfaces. □

Corollary 4.6. *Suppose ϕ_S is a generic sweep-out or partial sweep-out that essentially splits the sweep-out ϕ_T for N. Assume that the surface portions of the spines $\phi_s^{-1}(0)$ and $\phi_s^{-1}(1)$ are incompressible and boundary incompressible. Additionally, assume $d(T) \geq 3$. Then*

$$d(T) \leq 2 + \frac{8g(S) - 8 + 4|\partial S|}{|\partial S \cap \partial T|}.$$

Proof. Assume $d(T) \geq 3$. Suppose first that the closure of Q_a has nontrivial interception with the closure of Q_b. The closure must contain a vertex (t, s) of the graphic. Let $R_a \subset Q_a$ and $R_b \subset Q_b$ be the regions of the graphic that have (s, t) as a vertex. Then there is a path from any curve in R_a to any curve in R_b that goes through at most two edges of the graphic, as in Figure 5, and thus $d(T) \leq 2$.

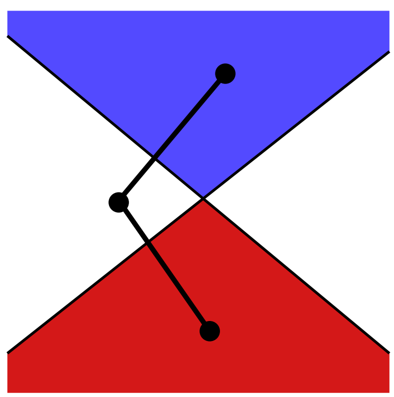

FIGURE 5. A distance two path in the graphic.

Suppose then that closure of Q_a is disjoint from the closure of Q_b. Since ϕ_S essentially splits ϕ_T, we can find an s such that $[0,1] \times \{s\}$ is disjoint from $Q_a \cup Q_b$ in $(0,1) \times (0,1)$. As the closure of Q_a is disjoint from the closure of Q_b, we can choose s so that the restriction of ϕ_T to S_s is Morse. By Lemma 4.5, we can find $u_s, v_s \in (0,1)$ such that the conditions of Theorem 3.1 are satisfied. Thus, the distance in the arc and curve complex for T from the arcs of $T_{u_s} \cap S_s$ to $T_{v_s} \cap S_s$ is at most $\frac{8g(S)-8+4|\partial S|}{|\partial S \cap \partial T|}$. As in the proof of Lemma 3.2, the conditions on u_s and v_s imply that the distance from $T_{u_s} \cap S_s$ to $\mathcal{D}_\downarrow$ and the distance from $T_{v_s} \cap S_s$ to $\mathcal{D}_\uparrow$ is each one. Therefore, the total distance is at most $2 + \frac{8g(S)-8+4|\partial S|}{|\partial S \cap \partial T|}$. $\square$

5. A panoply of distance bounds

Recall that N is a compact, connected, orientable 3-manifold with incompressible boundary. In order to apply Theorem 2.5, we make the further restriction that ∂N is a collection of tori. Let σ and τ be multislopes on $\partial_0 N$ such that $\Delta = \Delta(\sigma,\tau) \geq 1$. Let M_σ and M_τ be the result of filling components of $\partial_0 N$ using multislopes σ and τ respectively. Let L_σ and L_τ be the cores of the filling solid tori. Let S_σ and T_τ be bridge surfaces for (M_σ, L_σ) and (M_τ, L_τ), respectively, and let S and T be the induced σ-sloped and τ-sloped Heegaard surfaces for N. Assume that ∂S and ∂T intersect minimally. Before stating the theorem, we introduce a final definition. A *cabling annulus* for a one or two component link L in a 3-manifold is an essential annulus in the complement of L such that both boundary components of the annulus lie on L and neither is a meridian. The link L is said to be *cabled*.

Theorem 5.1. *Assume that L_σ is not an unknot and that T is not a planar surface with four or fewer boundary components. If $d(T) \geq 3$, then one of the following holds:*

(1) *S is σ-stabilized. If ∂N is connected then $d(T) \leq 2g(S)$.*
(2) *There is a cancellable σ'-sloped Heegaard surface for N with $\sigma' \subset \sigma$ non-empty. It has negative euler characteristic no greater than the negative euler characteristic of S. Let R be the cancelled bridge surface, and let ρ be the multi-slope of ∂R. If $\Delta(\rho,\tau) \geq 1$, we have:*
$$d(T) \leq 2 + \frac{8g(S) - 16 + 4|\partial S|}{|\partial R \cap \partial T|}.$$
In particular, if $\partial_0 N$ is a single torus and if $\Delta(\rho,\tau) \geq 1$, we have:
$$d(T) \leq 2 + \frac{2(g(S)-1)}{b(T)\Delta(\rho,\tau)}.$$
Furthermore, if ∂N is a single torus and if $\Delta(\rho,\tau) = 0$, then M_τ contains a closed incompressible surface of genus strictly less than the genus of S.
(3) *S is strongly boundary irreducible and*
$$d(T) \leq 2 + \frac{8g(S) - 8 + 4|\partial S|}{|\partial S \cap \partial T|}.$$
In particular, if ∂N is connected, then
$$d(T) \leq 2 + \frac{2}{b(T)\Delta}\left(\frac{g(S)-1}{b(S)} + 1\right).$$

(4) S is weakly boundary reducible, and one of the following occurs:

(a) There is a Heegaard surface for the exterior of L_σ with genus no more than $g(S)$. Furthermore, if ∂N is connected, then $d(T) \leq 2g(S)$.

(b) L_τ has an essential meridional annulus in its complement and a sublink of L_σ is cabled. Furthermore, ∂N is not connected.

(c) $d(T) \leq \max(3, 2g(S) + 2)$.

Furthermore, if ∂N is connected, then

$$d(T) \leq \max(3, 2 + \frac{2g(S)}{b(T)\Delta}).$$

Proof. By Theorem 2.5, the sloped Heegaard surface S can be untelescoped (possibly in zero steps) to a generalized sloped Heegaard splitting $(\mathcal{S}, \mathcal{F})$ such that every component of $\mathcal{F}$ is incompressible, no component of $N - (\mathcal{S} \cup \mathcal{F})$ is a trivial cobordism between a component of $\mathcal{S}$ and a component of $\mathcal{F}$, and each component of $\mathcal{S}$ in the complement of $\mathcal{F}$ is either

(a) slope stabilized, with slope a subset of σ
(b) strongly irreducible, but weakly boundary reducible
(c) strongly boundary irreducible.

We note also that S is weakly reducible if and only if $\mathcal{F} \neq \varnothing$. Although components of $\mathcal{F}$ are incompressible, they may be inessential. We first consider the cases (Case 1 and 2) when there is an *essential* thin surface, with the two cases corresponding to whether or not the essential thin surface has boundary. Subsequently, we may then assume that $\mathcal{F}$ consists only of *inessential* surfaces or is empty. We then single out a particular component S^* of $\mathcal{S}$. Case 3 deals with the case when S^* is closed and the remaining cases correspond to whether S^* satisfies (a), (b), or (c) from Theorem 2.5.

Case 1: There is an essential component of $\mathcal{F}$ without boundary.

Let F be a closed essential component of $\mathcal{F}$. If F is cut-compressible with respect to L_τ, then maximally cut-compress it and let F' be a component of the resulting surface. Suppose that F' is boundary parallel in $M(L_\tau)$. As F' has boundary it must be an annulus parallel to a segment of L_τ but in this case F was a torus parallel to a component of L_τ, a contradiction. Thus, F' is essential and cut-incompressible with respect to L_τ.

If F' is an annulus, then it is an essential meridional annulus in the complement of L_τ. If ∂N were connected, then by [2, Theorem 5.1] $d(T) \leq 2$. But this contradicts our hypothesis that $d(T) \geq 3$, and so ∂N is disconnected. In the complement of L_σ, the essential meridional

annulus for L_τ is a cabling annulus for a 1 or 2 component sublink of L_σ. This is conclusion (4b).

If F' is not an annulus, then by Theorem 3.3, $d(T) \leq 3$ or $d(T) \leq 2g(F') + |F' \cap L| \leq 2g(F) \leq 2g(S)$. This is conclusion (4a). $\square$(Case 1)

Case 2: There is an essential component of $\mathcal{F}$ with boundary.

Let F be an essential connected component of $\mathcal{F}$ with boundary. Since S is separating, and since $\mathcal{F}$ is obtained by compressing S, if $|\partial F| = 1$, there must exist another non-separating component $F' \neq F$ of $\mathcal{F}$ such that $\partial F' \neq \varnothing$ and $g(F) + g(F') \leq g(S)$. If such is the case, change notation to let F be the union of F and F'. In particular this means that $|\partial F| \geq 2$. Notice that F remains essential.

Each component of ∂F intersects a component of ∂T. The surface T has at least two boundary components on each component of $\partial_0 N$, so $|\partial F \cap \partial T| \geq 4$ and $|\partial F|/|\partial F \cap \partial T| \leq \frac{1}{2}$.

By Corollary 3.2,

$$d(T) \leq 2 + \frac{8g(F) - 8 + 4|\partial F|}{|\partial F \cap \partial T|}$$

Hence,

$$d(T) \leq 4 + \frac{8}{|\partial F \cap \partial T|}\big(g(F) - 1\big).$$

If $g(F) = 0$, then we have

$$d(T) \leq 4 - \frac{8}{|\partial F \cap \partial T|} < 4.$$

Since $d(T)$ is an integer, this implies $d(T) \leq 3$.

If $g(F) \geq 1$, then

$$d(T) \leq 4 + \frac{8g(F) - 8}{4} \leq 4 + 2g(F) - 2 = 2g(F) + 2$$

Since $g(F) \leq g(S)$, we have $d(T) \leq 2g(S) + 2$. This gives the first part of conclusion (4c).

If $\partial_0 N$ is connected, we can do better. Recall that $|\partial T| = 2b(T)$. Since $\partial_0 N$ is connected, ∂F has slope σ and so there are $2b(T)\Delta$ intersections between each component of ∂F and ∂T. Since $|\partial F| \geq 2$, this implies

$$|\partial F \cap \partial T| \geq 4b(T)\Delta.$$

and

$$|\partial F|/|\partial F \cap \partial T| = \frac{1}{2b(T)\Delta}.$$

Thus, in this case, we either have $d(T) \leq 3$ (as before) or

$$d(T) \leq 2 + \frac{2}{b(T)\Delta}\big(g(S) - 1\big) + \frac{2}{b(T)\Delta} = 2 + \frac{2}{b(T)\Delta} g(S).$$

This is the remainder of conclusion (4c). $\square$(Case 2)

By Cases (1) and (2), we may assume that either $\mathcal{F} = \varnothing$ or that every component of $\mathcal{F}$ is inessential. Notice that, by the definition of generalized sloped Heegaard splitting, any component of $N - \mathcal{F}$ adjacent to a component of $\mathcal{F}$ with boundary contains a component of $\mathcal{S}$ with boundary. The 3-manifold $N - \mathcal{F}$ has a component that is homeomorphic to N (possibly after capping off 2-spheres in the boundary with 3-balls). Let N' be the 3-manifold resulting from capping off the 2-spheres in the boundary. It contains a Heegaard surface $S^* \subset \mathcal{S}$ with boundary a subset of σ. Of course, if $\mathcal{F} = \varnothing$, then $S^* = S$. If no component of $\mathcal{F}$ is an inessential 2-sphere, then N' is a submanifold of N. Let σ' be the slope of ∂S^*. Since $\Delta \geq 1$ and since $\sigma' \subset \sigma$, either $\sigma' = \varnothing$ or $\Delta(\sigma', \tau) \geq \Delta \geq 1$.

If S^* were an annulus, then L_σ could be isotoped in M_σ to be in one bridge position with respect to the sphere obtained by capping off ∂S^* with disks. This implies that L_σ is the unknot, a contradiction. We may, therefore, assume that S^* is not an annulus.

Case 3: S^* is closed. I.e. $\sigma' = \varnothing$.

Since S is not closed, $\mathcal{F} \neq \varnothing$. The homeomorphism of N' to N takes S^* to a Heegaard surface for the exterior of L_σ. Since S^* is obtained by compressing S, $g(S^*) \leq g(S)$. If ∂N is connected, by Theorem 3.4, $d(T) \leq 2g(S^*) \leq 2g(S)$. This is conclusion (4a). $\square$(Case 3)

Case 4: S^* is σ'-stabilized.

If a component of $\mathcal{S}$ is σ-stabilized, then the corresponding bridge surface is removable and vice versa (Lemma 2.3). Hence, by Theorem 6.3 of [46], if S^* is σ'-stabilized, then S is σ-stabilized. If ∂N is connected, this implies that S_σ is isotopic to a Heegaard surface for N. By Theorem 3.4, it follows that $d(T) \leq 2g(S)$. This is conclusion (1). $\square$(Case 4)

Henceforth, we assume that S^* has boundary (i.e. $\sigma' \neq \varnothing$) and that S^* and S are not slope-stabilized. Consequently, S^* is strongly irreducible, although it may be weakly boundary reducible.

Case 5: S^* is weakly boundary reducible.

The homeomorphism of N' to N takes S^* to a sloped Heegaard surface for N, which we continue to call S^*. Let $\overline{S}^*$ be the corresponding bridge surface for the (3-manifold, link) pair obtained by filling boundary components of N with slope σ'. Since S^* is not an annulus, the

frontier of a regular neighborhood of a bridge disk for $\overline{S}^*$ is a compressing disk for S^*. Since S^* is strongly irreducible, $\overline{S}^*$ must be perturbed or cancellable. Since $\overline{S}^*$ is strongly irreducible, by Lemma 2.1, it is cancellable. Let R be the cancelled bridge surface and let ρ be its boundary slope. The surface R is essential in N and, if ∂N is connected, after capping off ∂R with disks, remains essential in the manifold obtained by filling ∂N by slope ρ. If R is connected, then it has genus one less than the genus of S^*. If R is disconnected, then the sum of the genera of its components is equal to the genus of S^*. Thus, if $\Delta(\rho, \tau) = 0$, then $\rho = \tau$ and we have conclusion (2).

Otherwise, if $\Delta(\rho, \tau) \geq 1$, we can apply Corollary 3.2 to R to conclude

$$d(T) \leq 2 + \frac{8g(R) - 8|R| + 4|\partial R|}{|\partial R \cap \partial T|}$$

Suppose R is connected. Then $g(R) = g(S^*) - 1$, $|R| = 1$, and $|\partial R| = |\partial S^*|$, so that

$$d(T) \leq 2 + \frac{8g(S^*) - 16 + 4|\partial S^*|}{|\partial R \cap \partial T|}.$$

Suppose R is disconnected. Then $g(R) = g(S^*)$ (where the genus of a disconnected surface is the sum of the genera of the components), $|R| = 2$ and $|\partial R| = |\partial S^*|$ so that

$$d(T) \leq 2 + \frac{8g(S^*) - 16 + 4|\partial S^*|}{|\partial R \cap \partial T|}.$$

Notice that we achieve the same bound on $d(T)$ if R is connected or not. The surface S^* is obtained by compressing S and so $-\chi(S^*) \leq -\chi(S)$. This implies,

$$8g(S^*) - 16 + 4|\partial S^*| \leq 8g(S) - 16 + 4|\partial S|.$$

Consequently,

$$d(T) \leq 2 + \frac{8g(S) - 16 + 4|\partial S|}{|\partial R \cap \partial T|},$$

which is the first part of conclusion (2).

If furthermore $\partial_0 N$ is connected, we have $|\partial S^*| = 2$. Consequently, $|\partial R \cap \partial T| = 4b(T)\Delta(\rho, \tau)$ implying

$$d(T) \leq 2 + \frac{2(g(S^*) - 1)}{b(T)\Delta(\rho, \tau)}.$$

Since $g(S^*) \leq g(S)$, this is the remainder of conclusion (2). $\square$(Case 5)

Case 6: S^* is strongly boundary irreducible.

Under the homeomorphism from N' to N, the surface S^* is taken to a σ'-sloped Heegaard surface for N. We continue to call that surface S^*. Let ϕ_{S^*} be a sweepout of N given by S^* and let ϕ_T be a sweepout of N given by T. Recall the definition of Q_a, Q_b and (t, s) from Remark 4.1. By Remark 4.1, there exists t_0 such that for ϵ sufficiently small, (t_0, ϵ) is contained in Q_a. Similarly, there exists t_1 such that for ϵ sufficiently small, $(t_1, 1-\epsilon)$ is contained in Q_b. Label each $s \in (0, 1)$ with a if $[0, 1] \times \{s\}$ meets Q_a and label it with b if $[0, 1] \times \{s\}$ meets Q_b. Since all s near zero receive the label a and all s near 1 receive the label b, then either there exists an s_0 such that s_0 is not labeled a and is not labeled b (i.e. ϕ_{S^*} essentially splits ϕ_T) or there exists an s_0 such that s_0 is labeled both a and b (i.e. ϕ_{S^*} essentially spans ϕ_T).

By assumption, S^* is not weakly boundary reducible and S^* is not an annulus, so, by Theorem 4.3, ϕ_{S^*} cannot essentially span ϕ_T. Hence, ϕ_{S^*} must essentially split ϕ_T.

By Corollary 4.6, we have

$$d(T) \leq 2 + \frac{8g(S^*) - 8 + 4|\partial S^*|}{|\partial S^* \cap \partial T|}.$$

If $\partial_0 N$ is connected, then $|\partial S^* \cap \partial T| = 4b(S^*)b(T)\Delta(\sigma', \tau)$. Thus, if $S^* = S$, we have conclusion (3).

Suppose that $S^* \neq S$. The surface S^* is separating, as is the surface T. Thus, $|\partial S^* \cap \partial T| \geq 4$. Furthermore, each component of ∂S^* intersects ∂T at least two times, so $|\partial S^*|/|\partial S^* \cap \partial T| \leq 1/2$. Thus, if $g(S^*) \geq 1$, we obtain

$$d(T) \leq 2g(S^*) + 2 \leq 2g(S) + 2.$$

If $g(S^*) = 0$, we have:

$$d(T) < 4.$$

Since $d(T)$ is an integer, this implies $d(T) \leq 3$. This is the first part of conclusion (4c).

If $\partial_0 N$ is connected, then we can do better. For in that case, $\sigma' = \sigma$, $|\partial S^* \cap \partial T| \geq 4b(T)\Delta$ and $|\partial S^*|/|\partial S^* \cap \partial T| \leq 1/2b(T)\Delta$. If $g(S^*) = 0$, we still get $d(T) \leq 3$. But if $g(S^*) \geq 1$, we obtain the inequality

$$d(T) \leq 2 + \frac{2g(S^*)}{b(T)\Delta} \leq 2 + \frac{2g(S)}{b(T)\Delta}$$

which is the remainder of conclusion (4c). □(Case 6) □

6. Exceptional Surgeries on Knots

Theorem 6.1. *Let M be a closed, compact, orientable manifold and suppose that $L \subset M$ is a knot in bridge position with respect to a*

Heegaard surface T_τ for M. Assume that the boundary of a regular neighborhood of L is incompressible in the exterior of L and that if T_τ is a sphere, then $|L \cap T_\tau| \geq 6$. Then:

(1) If $M = S^3$, $T_\tau = S^2$, and surgery on L produces a reducible 3-manifold, then $d(T_\tau) \leq 2$.
(2) If $M = S^3$, $T_\tau = S^2$, and surgery on L produces a 3-manifold with an essential torus, then $d(T_\tau) \leq 2$.
(3) If $M = S^3$, $T_\tau = S^2$, L and surgery on L produces a lens space, then $d(T_\tau) \leq 3$.
(4) If $M = S^3$, $T_\tau = S^2$, and surgery on L produces a small Seifert fibered space other than S^3 or a lens space, then $d(T_\tau) \leq 4$.
(5) If M does not contain an essential sphere or torus and if L is a hyperbolic knot with a non-trivial non-hyperbolic surgery then $d(T_\tau) \leq 6$.
(6) If $M \neq S^3$ and a non-trivial surgery on L produces a 3-manifold with Heegaard genus no larger than that of M, then

$$d(T_\tau) \leq max(4, 2g(T_\tau) + 2).$$

Similar results for links follow from Theorem 5.1. We leave the calculations of specific bounds in these cases to the reader.

Proof. Let $N = M(L)$ and choose a surgery slope $\sigma \neq \tau$ on ∂N. Then $\Delta(\sigma, \tau) \geq 1$. Let $M_\tau = M$, $L_\tau = L$ and let M_σ be the result of filling ∂N with slope σ. Let L_σ be the core of the surgery solid torus. Let $T = T_\tau \cap N$. Given a surface S_σ in M_σ transverse to L_σ, let $S = S_\sigma \cap N$. Assume that ∂S and ∂T intersect minimally. Assume that $d(T_\tau) = d(T) \geq 2$. Let $\Delta = \Delta(\sigma, \tau)$.

Since ∂N is incompressible, L_σ cannot be the unknot in M_σ.

Proof of (1) Assume that M_σ is reducible and let S_σ be an essential 2–sphere intersecting L_σ minimally out of all such 2-spheres. Then S is an essential planar surface in N. Since $g(S) = 0$, by Corollary 3.2

$$d(T) \leq 2 + \frac{4|\partial S| - 8}{|\partial S \cap \partial T|}.$$

Since, $|\partial S \cap \partial T| = 2b(T)|\partial S|\Delta$, we obtain:

$$d(T) \leq 2 + \frac{2}{b(T)\Delta}\left(1 - \frac{2}{|\partial S|}\right).$$

Since N is not a solid torus, $|\partial S| \geq 2$ and $b(T) \geq 3$. Consequently,

$$d(T) \leq 2 + \frac{2}{3}\left(1 - \frac{2}{|\partial S|}\right) < 3.$$

Since $d(T)$ is an integer, $d(T) \leq 2$.

Proof of (2)

Suppose that M_σ contains an essential torus. Let S_σ be such a torus that, out of all such, intersects L_σ minimally. Then S is an essential, possibly punctured, torus in N.

Case 1: $S_\sigma \cap L_\sigma = \emptyset$.

If S_σ is cut-compressible, then L_σ has a decomposing sphere. In this case by Corollary 3.2 and by the arithmetic from (1), $d(T) \leq 2$.

If S_σ is not cut-compressible, then it is c-essential and by Theorem 3.3, $d(T) \leq 2$ since $g(S) = 1$.

Case 2: $|S_\sigma \cap L_\sigma| \geq 1$. Because $g(S) = 1$, Corollary 3.2 implies that

$$d(T) \leq 2 + \frac{4|\partial S|}{|\partial S \cap \partial T|}.$$

Since $|\partial S \cap \partial T| = 2b(T)|\partial S|\Delta$,

$$d(T) \leq 2 + \frac{2}{b(T)\Delta}.$$

By assumption, $b(T) \geq 3$ so $d(T) \leq 2 + \frac{2}{3\Delta}$. Since $d(T)$ is an integer, $d(T) \leq 2$.

Proof of (3) Suppose M_σ is a lens space and choose a genus 1 bridge surface S_σ for (M_σ, L_σ) such that L_σ is in minimum bridge position with respect to S_σ. Then S is not σ-stabilized. Since $M_\tau = S^3$, M_τ does not contain an essential sphere or torus. If $d(T) \geq 3$, then by Theorem 5.1, taking into account that $g(S) = 1$ and ∂N is a single torus, either:

- $d(T) = 3$, or
- $d(T) \leq 2 + \frac{2}{b(T)\Delta}$.

Since $T_\tau = S^2$, by hypothesis $b(T) \geq 3$. Thus, if the second possibility holds, $d(T) \leq 8/3$. Since $d(T)$ is an integer, $d(T) \leq 2$.

Proof of (4) Let M_σ be a small Seifert fibered space. It has Heegaard genus no more than 2 by [7], so let S_σ be a bridge surface for (M_σ, L_σ), chosen so as to minimize the pair $(g(S), b(S))$ lexicographically. $M_\tau = S^3$ does not contain an essential sphere or torus.

Assume that $d(T) \geq 3$. By Theorem 5.1, one of the following occurs (the numbering has been chosen to match that of the theorem).

(1) $d(T) \leq 4$.
(2) $d(T) \leq 2 + \frac{2}{b(T)\Delta(\rho,\tau)}$.
(3) $d(T) \leq 2 + \frac{2}{b(T)\Delta}\left(\frac{1}{b(S)} + 1\right)$.
(4) $d(T) \leq \max(4, 2 + \frac{4}{b(T)\Delta})$.

Since $b(T) \geq 3$, in the second case, then $d(T) \leq 2 + \frac{2}{3}$ and, since $d(T)$ is an integer, we conclude $d(T) \leq 2$. In the third case, we obtain

$d(T) \leq 2 + 4/3$ and, since $d(T)$ is an integer, $d(T) \leq 3$. Similarly, the fourth case implies $d(T) \leq 4$.

Proof of (5) Suppose that M_σ is non-hyperbolic. By the geometrization theorem it is reducible, toroidal, or a small Seifert fibered space. Each case is very similar to the previous parts. The only major change is that we are only guaranteed that $b(T) \geq 1$, not that $b(T) \geq 3$.

Case 5a: Suppose M_σ is reducible. In this case, choose S_σ to be an essential 2–sphere minimally intersecting L_σ, as before. We get $d(T) \leq 4$.

Case 5b: Suppose M_σ is toroidal. In this case, choose S_σ to be an essential torus minimally intersecting L_σ, as before. We get $d(T) \leq 4$.

Case 5c: Suppose M_σ is a lens space. In this case, S_σ is a torus and we get $d(T) \leq 4$.

Case 5d: Suppose M_σ is a small Seifert fibered space. In this case, let S_σ be a minimal genus Heegaard surface. It has genus no more than 2. Using the fact that M_τ does not have an essential sphere or torus, we obtain $d(T) \leq 6$.

Proof of (6) Assume that M_σ has Heegaard genus no larger than that of M_τ. Let S_σ be a Heegaard surface for M_σ with $g(S_\sigma) \leq g(T_\tau)$. Isotope it so that it is a minimal bridge surface for (M_σ, L_σ). Assume $d(T_\tau) \geq 3$. Apply Theorem 5.1 and consider each possible conclusion. Note that ∂N is connected.

If S is σ-stabilized, then $d(T) \leq 2g(S) \leq 2g(T)$.

Suppose that S is strongly boundary irreducible and that S_σ is cancellable with $\Delta(\rho, \tau) \geq 1$. If S_σ is a 2–sphere, then $d(T) < 2$, a contradiction. If $g(S_\sigma) \geq 1$, then $d(T) \leq 2g(S) \leq 2g(T)$.

Suppose that S is strongly boundary irreducible and that S_σ is cancellable with $\Delta(\rho, \tau) = 0$. Let R be the associated cancelled bridge surface with boundary slope $\rho = \tau$. The surface R is essential and has 2-boundary components. As in Case (4c) of the proof of Theorem 5.1, maximally cut compress R to obtain a surface F'. Let $R = R_0, R_1, \ldots, R_n$ be the sequence of cut compressions. We note that cut compressing cannot create compressing disks and so each surface R_i remains incompressible. If R_i has a component that is boundary-parallel, then R_{i-1} had a component that was boundary parallel or compressible. Hence, R_n is both essential and cut-incompressible. If some component of R_n is an annulus then $d(T) \leq 4$ (as in the proof of Case 5a above). Otherwise, we may apply Theorem 3.3. By that theorem, $d(T) \leq \max(3, -\chi(R^*) + 2)$ for any component R^* of R_n. Since each cut compression leaves negative euler characteristic unchanged and since no component of R^* has negative euler characteristic, we

have $d(T) \leq 3$ or

$$d(T) \leq -\chi(R) + 2 = 2g(R) - 2|R| + 4 \leq 2g(R) + 2 \leq 2g(T) + 2.$$

Suppose that S is strongly boundary irreducible. If $g(S) = 0$, then $d(T) \leq 3$. Otherwise,

$$d(T) \leq 2 + 2g(S) \leq 2 + 2g(T).$$

If S is weakly boundary reducible and if $d(T) > 3$, then, once again, $d(T) \leq 2 + 2g(S) \leq 2 + 2g(T)$. □

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